THE BOWDITCH BOUNDARY OF $(G, \mathcal{H})$ WHEN G IS HYPERBOLIC

JASON FOX MANNING

ABSTRACT. In this note we use Yaman's dynamical characterization of relative hyperbolicity to prove a theorem of Bowditch about relatively hyperbolic pairs $(G, \mathcal{H})$ with G hyperbolic. Our proof additionally gives a description of the Bowditch boundary of such a pair.

1. Introduction

Let G be a group. A collection $\mathcal{H} = \{H_1, \ldots, H_n\}$ of subgroups of G is said to be *almost malnormal* if every infinite intersection of the form $H_i \cap g^{-1} H_j g$ satisfies both $i = j$ and $g \in H_i$.

In an extremely influential paper from 1999, recently published in *IJAC* [Bow12], Bowditch proves the following useful theorem:

Theorem 1.1. [Bow12, Theorem 7.11] *Let G be a nonelementary hyperbolic group, and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be an almost malnormal collection of proper, quasiconvex subgroups of G. Then G is hyperbolic relative to $\mathcal{H}$.*

Remark 1.2. The converse to this theorem also holds. If $(G, \mathcal{H})$ is any relatively hyperbolic pair, then the collection $\mathcal{H}$ is almost malnormal by [Osi06, Proposition 2.36]. Moreover the elements of $\mathcal{H}$ are undistorted in G [Osi06, Lemma 5.4]. Undistorted subgroups of a hyperbolic group are quasiconvex.

In this note, we give a proof of Theorem 1.1 which differs from Bowditch's. The strategy we follow is to exploit the dynamical characterization of relative hyperbolicity given by Yaman in [Yam04]. By doing so, we are able to obtain some more information about the pair $(G, \mathcal{H})$. In particular, we obtain an explicit description of its Bowditch boundary $\partial(G, \mathcal{H})$. (This same strategy was applied by Dahmani to describe the boundary of certain amalgams of relatively hyperbolic groups in [Dah03].) Let ∂G be the Gromov boundary of the group G. If H is quasiconvex in a hyperbolic group G, its limit set $\Lambda(H) \subset \partial G$ is homeomorphic to the Gromov boundary ∂H of H. The next theorem says that $\partial(G, \mathcal{H})$ is obtained by smashing the limit sets of gHg^{-1} to points, for $H \in \mathcal{H}$ and $g \in G$.

Theorem 1.3. [1] *Let G be hyperbolic, and let $\mathcal{H}$ be an almost malnormal collection of infinite quasi-convex proper subgroups of G. Let $\mathcal{L}$ be the set of G–translates of limit sets of elements of $\mathcal{H}$. The Bowditch boundary $\partial(G, \mathcal{H})$ is obtained from the Gromov boundary ∂G as a decomposition space $\partial G/\mathcal{L}$.*

[1] Since I posted this note, it's been pointed out to me that this theorem too was already well-known. See in particular Tran [Tra13] for an alternate proof which additionally gives the Bowditch boundary of $(G, \mathcal{P})$ when G is CAT(0). Tran also points out previous results of Gerasimov and Gerasimov–Potyagailo [Ger12, GP09], or alternatively Matsuda–Oguni–Yamagata [MOY12] which can be used to give other proofs of Theorem 1.3.

In particular, we can bound the dimension of this space:

Corollary 1.4. *Let G be a hyperbolic group and $\mathcal{H}$ a malnormal collection of infinite quasi-convex proper subgroups. Then* $\dim \partial(G,\mathcal{H}) \leq \dim \partial G + 1$.

Proof. This follows from the Addition Theorem of dimension theory, a special case of which says that if a compact metric space $M = A \cup B$, then $\dim(M) \leq \dim(A) + \dim(B) + 1$. By Theorem 1.3, $\partial(G,\mathcal{H})$ can be written as a disjoint union of a countable set (coming from the limit sets of the conjugates of the elements of $\mathcal{H}$) with a subset of ∂G. □

At least conjecturally, this proposition gives cohomological information about the pair:

Conjecture 1.5. *Let G be torsion-free and hyperbolic relative to $\mathcal{H}$. Let* $\mathrm{cd}(G,\mathcal{H})$ *be the cohomological dimension of the pair $(G,\mathcal{H})$, and let* $\dim$ *be topological dimension. Then*

$$\mathrm{cd}(G,\mathcal{H}) = \dim \partial(G,\mathcal{H}) + 1, \tag{1}$$

and more generally,

$$H^q(G,\mathcal{H};\mathbb{Z}G) = \check{H}^{q-1}(\partial(G,\mathcal{H})) \tag{2}$$

for all integers q.

In the absolute setting ($\mathcal{H} = \emptyset$), Equations (1) and (2) are results of Bestvina–Mess [BM91]. In case G is a geometrically finite group of isometries of $\mathbb{H}^n$ for some n and $\mathcal{H}$ is the collection (up to conjugacy) of maximal parabolic subgroups of G, Kapovich establishes equations (1) and (2) in [Kap09, Proposition 9.6], and remarks that the proof should extend easily to the case in which all elements of $\mathcal{H}$ are virtually nilpotent. The key step which must be generalized is the existence of an appropriate space for which the Bowditch boundary is a $\mathcal{Z}$–set.[2]

Definition 1.6. Suppose that M is a compact metrizable space with at least 3 points, and let G act on M by homeomorphisms. The action is a *convergence group action* if the induced action on the space $\Theta^3(M)$ of unordered triples of distinct points in M is properly discontinuous.

An element $g \in G$ is *loxodromic* if it has infinite order and fixes exactly two points of M.

A point $p \in M$ is a *bounded parabolic point* if $\mathrm{Stab}_G(p)$ contains no loxodromics, and acts cocompactly on $M \setminus \{p\}$.

A point $p \in M$ is a *conical limit point* if there is a sequence $\{g_i\}$ in G and a pair of points $a \neq b$ in M so that:

(1) $\lim_{i\to\infty} g_i(p) = a$, and
(2) $\lim_{i\to\infty} g_i(x) = b$ for all $x \in M \setminus \{p\}$.

A convergence group action of G on M is *geometrically finite* if every point in M is either a bounded parabolic point or a conical limit point.

[2] I believe that (an interpretation of) Kapovich's proof actually extends to the case in which all peripheral groups have finite $K(\pi,1)$'s. So the conjecture is almost certainly true with that hypothesis. It may not be true in greater generality. I don't know.

Remark 1.7. If G is countable, G acts on M as a convergence group, and there is no closed G-invariant proper subset of M, then M is separable. In particular ∂G for G hyperbolic is separable, as is any space admitting a geometrically finite convergence group action by a countable group.

Bowditch proved in [Bow98] that if G acts on M as a convergence group and every point of M is a conical limit point, then G is hyperbolic. Conversely, if G is hyperbolic, then G acts as a convergence group on ∂G, and every point in ∂G is a conical limit point. For general geometrically finite actions, we have the following result of Yaman:

Theorem 1.8. [Yam04, Theorem 0.1] *Suppose that M is a non-empty perfect metrizable compact space, and suppose that G acts on M as a geometrically finite convergence group. Let $B \subset M$ be the set of bounded parabolic points. Let $\{p_1, \ldots, p_n\}$ be a set of orbit representatives for the action of G on B. For each i let P_i be the stabilizer in G of p_i, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$.*

G is relatively hyperbolic, relative to $\mathcal{P}$.

Outline of proof of Theorem 1.1. We prove Theorem 1.1 by constructing a space M on which G acts as a geometrically finite convergence group, so that the parabolic point stabilizers are all conjugate to elements of $\mathcal{H}$. The space M is a quotient of ∂G, constructed as follows. The hypotheses on $\mathcal{H}$ imply that the boundaries ∂H_i embed in ∂G for each i, and that $g\partial H_i \cap h\partial H_j$ is empty unless $i = j$ and $g^{-1}h \in H_i$. Let

$$A = \{g\partial H_i \mid g \in G, \text{ and } H_i \in \mathcal{H}\},$$

and let

$$B = \{\{x\} \mid x \in \partial G \setminus \bigcup A\}.$$

The union $\mathcal{C} = A \cup B$ is therefore a decomposition of ∂G into closed sets. We let M be the quotient topological space $\partial G/\mathcal{C} = A \cup B$. There is clearly an action of G on M by homeomorphisms.

We now have a sequence of four claims, which we prove later.

Claim 1. *$M = A \cup B$ is a perfect metrizable space.*

Claim 2. *G acts as a convergence group on M.*

Claim 3. *For $x \in A$, x is a bounded parabolic point, with stabilizer conjugate to an element of $\mathcal{H}$.*

Claim 4. *For $x \in B$, x is a conical limit point.*

Given the claims, we may apply Yaman's theorem 1.8 to conclude that G is relatively hyperbolic, relative to $\mathcal{H}$. □

2. Proofs of claims

In what follows we fix some δ-hyperbolic Cayley graph Γ of G. We'll use the notation $a \mapsto \bar{a}$ for the map from ∂G to the decomposition space M.

2.1. **Claim 1.** We're going to need some basic point-set topology. What we need is in Hocking and Young [HY88], mostly Chapter 2, Section 16, and Chapter 5, Section 6.

Definition 2.1. Given a sequence $\{D_i\}$ of subsets of a topological space X, the lim inf and lim sup of $\{D_i\}$ are defined to be

$$\liminf D_i = \{x \in X \mid \text{for all open } U \ni x,\, U \cap D_i \neq \emptyset \text{ for almost all } i\}$$

and

$$\limsup D_i = \{x \in X \mid \text{for all open } U \ni x,\, U \cap D_i \neq \emptyset \text{ for infinitely many } i\}$$

The notion of *upper semicontinuity* for a decomposition of a compact metric space into closed sets can be phrased in terms of Definition 2.1. The following can be extracted from [HY88, section 3–6] and standard metrization theorems.

Proposition 2.2. *Let X be a compact separable metric space, and let $\mathcal{D}$ be a decomposition of X into disjoint closed sets. Let Y be the quotient of X obtained by identifying each element of $\mathcal{D}$ to a point. The following are equivalent:*

(1) *$\mathcal{D}$ is upper semicontinuous.*
(2) *Y is a compact metric space.*
(3) *For any sequence $\{D_i\}$ of elements of $\mathcal{D}$, and any $D \in \mathcal{D}$ so that $D \cap \liminf D_i$ is nonempty, we have $\limsup D_i \subset D$.*

Lemma 2.3. *Let $\{C_i\}$ be a sequence of elements of the decomposition $\mathcal{C} = A \cup B$ of ∂G, so that no element appears infinitely many times. If $\liminf C_i \neq \emptyset$, then $\limsup C_i = \liminf C_i$ is a single point.*

Proof. By way of contradiction, assume there are two points in $\liminf C_i$. There are therefore points $a_i \in C_i$ limiting on x, and $b_i \in C_i$ limiting on y. It follows that the C_i must eventually be of the form $g_i \partial H_{j_i}$, for $H_{j_i} \in \mathcal{H}$. Passing to a subsequence (which can only make the lim inf bigger) we may assume all the $H_{j_i} = H$ for some fixed λ-quasiconvex subgroup H.

In a proper δ-hyperbolic geodesic space, geodesics between arbitrary points at infinity exist, and triangles formed from such geodesics are 3δ–thin. It follows that a geodesic between limit points of a λ-quasiconvex set lies within $\lambda + 6\delta$ of the quasiconvex set. Let p be a point on a geodesic from x to y, and let γ_i be a geodesic from a_i to b_i. For large enough i the geodesic γ_i passes within 6δ of p, so the sets $g_i H$ must, for large enough i intersect the $(\lambda + 12\delta)$–ball about p. Since this ball is finite and the cosets of H are disjoint, some $g_i H$ must appear infinitely often, contradicting the assumption that no C_i appears infinitely many times.

A similar argument shows that in case $\liminf C_i$ is a single point, then $\limsup C_i$ cannot be any larger than $\liminf C_i$. □

Remark 2.4. If $\liminf C_i$ is empty, then $\limsup\{C_d\}$ can be any closed subset of ∂G.

Proof of Claim 1. We first verify condition (3) of Proposition 2.2. Let $\{C_i\}$ be some sequence of elements of the decomposition $\mathcal{C}$, and let D be an element of the decomposition so that $D \cap \liminf C_i \neq \emptyset$. If no element of $\mathcal{C}$ appears infinitely many times in $\{C_i\}$, then Lemma 2.3 implies that $\liminf C_i = \limsup D_i$ is a single point, so (3) is satisfied almost trivially. We can therefore assume that for some $C \in \mathcal{C}$, there are infinitely many i for which $C_i = C$.

In fact, there can only be one such C, for otherwise we would have $\liminf C_i = \emptyset$. If all but finitely many C_i satisfy $C_i = C$, then $\liminf C_i = \limsup C_i = C$, and it is easy to see that condition (3) is satisfied.

We may therefore assume that $C_i \neq C$ for infinitely many i. Let $\{B_i\}$ be the sequence made up of those $C_i \neq C$. No B_i appears more than finitely many times. Since $\{B_i\}$ is a subsequence of $\{C_i\}$, we have

$$\liminf C_i \subseteq \liminf B_i \subseteq \limsup B_i \subseteq \limsup C_i.$$

Applying Lemma 2.3 to $\{B_i\}$, we deduce that $\liminf B_i = \limsup B_i$ is a single point. It follows that $\liminf C_i$ is a single point, and so condition (3) is again satisfied trivially.

We've shown that $M = \partial G/\mathcal{C}$ is a compact metric space. We now show M is perfect. Let $p \in M$.

Suppose first that $p \in B$, i.e., that the preimage in ∂G is a single point $\tilde{p}$. Because G is nonelementary, ∂G is perfect. Thus there is a sequence of points $x_i \in \partial G \setminus \{\tilde{p}\}$ limiting on p. The image of this sequence limits on p.

Now suppose that $p \in A$, i.e., the preimage of p in ∂G is equal to $g\partial H$ for some $g \in G$ and some $H \in \mathcal{H}$. Choose any point $x \in \partial G \setminus \partial H$, and any infinite order element h of gHg^{-1}. The points $h^i x$ project to distinct points in $M \setminus \{p\}$, limiting on p. □

2.2. **Claim 2.** In [Bow99], Bowditch gives a characterization of convergence group actions in terms of *collapsing sets*. We rephrase Bowditch slightly in what follows.

Definition 2.5. Let G act by homeomorphisms on M. Suppose that $\{g_i\}$ is a sequence of distinct elements of G. Suppose that there exist points a and b (called the *attracting* and *repelling* points, respectively) so that whenever $K \subseteq M \setminus \{a\}$ and $L \subseteq M \setminus \{b\}$, the set $\{i \mid g_i K \cap L \neq \emptyset\}$ is finite. Then $\{g_i\}$ is a *collapsing sequence.*

Proposition 2.6. [Bow99, Proposition 1.1] *Let G, a countable group, act on M, a compact Hausdorff space with at least 3 points. Then G acts as a convergence group if and only if every infinite sequence in G contains a subsequence which is collapsing.*

Proof of Claim 2. We use the characterization of 2.6. Let $\{\gamma_i\}$ be an infinite sequence in G. Since the action of G on ∂G is convergence, there is a collapsing subsequence $\{g_i\}$ of $\{\gamma_i\}$; i.e., there are points a and b in ∂G which are attracting and repelling in the sense of Definition 2.5. We will show that $\{g_i\}$ is also a collapsing sequence for the action of G on M, and that the images $\bar{a}$ and $\bar{b}$ in M are the attracting and repelling points for this sequence.

Let $K \subseteq M \setminus \{\bar{a}\}$ and $L \subseteq M \setminus \{\bar{b}\}$ be compact sets, and let $\tilde{K}$ and $\tilde{L}$ be the preimages of K and L in ∂G. We have $\tilde{K} \subseteq \partial G \setminus \{a\}$ and $\tilde{L} \subseteq \partial G \setminus \{b\}$, so $\{i \mid g_i\tilde{K} \cap \tilde{L} \neq \emptyset\}$ is finite. But for each i, $g_i K \cap L = \cdots = \pi(g_i\tilde{K} \cap \tilde{L})$, so $\{i \mid g_i K \cap L\}$ is also finite. □

Remark 2.7. In the preceding proof it is possible for a and b to be distinct, but $\bar{a} = \bar{b}$.

2.3. **Claim 3.**

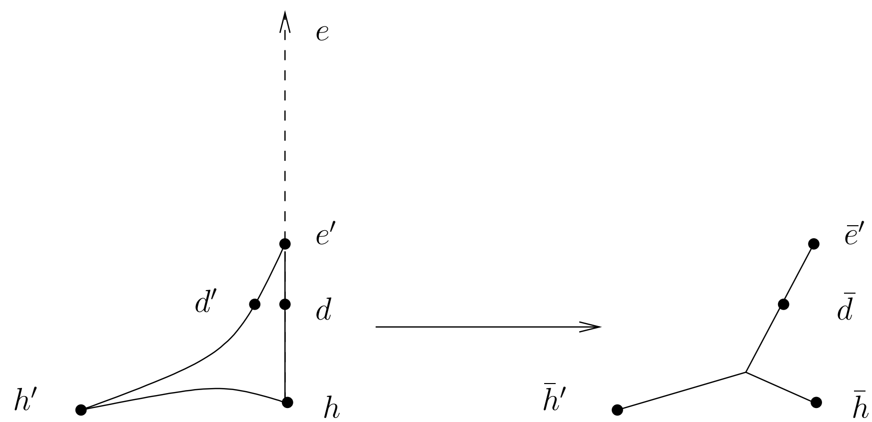

FIGURE 1. Bounding the distance from h to d.

Proof of Claim 3. Let $p \in A \subseteq M$ be the image of $g\partial H$ for $g \in G$ and $H \in \mathcal{H}$. Let $P = gHg^{-1}$. Since H is equal to its own commensurator, so is P, and $P = \mathrm{Stab}_G(p)$. We must show that P acts cocompactly on $M \setminus \{p\}$. The subgroup P is λ-quasiconvex in Γ (the Cayley graph of G) for some $\lambda > 0$. Let N be a closed R-neighborhood of P in Γ for some large integer R, with $R > 2\lambda + 10\delta$. Note that any geodesic from 1 to a point in ∂H stays inside N, and any geodesic from 1 to a point in $\partial G \setminus \partial P$ eventually leaves N. Write $\mathrm{Front}(N)$ for the frontier of N.

Let $C = \{g \in \mathrm{Front}(N) \mid d(g, 1) \leq 2R + 100\delta\}$. Let E be the set of points $e \in \partial X$ so that there is a geodesic from 1 to e passing through C. The set E is compact, and lies entirely in $\partial G \setminus \partial P$. We will show that $PE = \partial G \setminus \partial P$. Let $e \in \partial G \setminus \partial P$, and let $h \in P$ be "coarsely closest" to e in the following sense: If $\{x_i\}$ is a sequence of points in X tending to e, then for large enough i, we have, for any $h' \in P$, $d(h, e_i) \leq d(h', e_i) + 4\delta$. Let γ be a geodesic ray from h to e, and let d be the unique point in $\gamma \cap \mathrm{Front}(N)$. Since $d \in \mathrm{Front}(N)$, there is some h' so that $d(h', d) = R$. Let e' be a point on γ so that $10R < d(h, e') \leq d(h', e') + 4\delta$, and consider a geodesic triangle made up of that part of γ between h and e', some geodesic between h' and h, and some geodesic between h' and e'. This triangle has a corresponding comparison tripod, as in Figure 1. Since any geodesic from h' to h must stay $R - \lambda > \delta$ away from $\mathrm{Front}(N)$, the point $\bar{d}$ must lie on the leg of the tripod corresponding to e'. Let d' be the point on the geodesic from h' to e' which projects to $\bar{d}$ in the comparison tripod. Since $d(h', d) = R$, $d(h', d') \leq R + \delta$. Now notice that

$$\begin{aligned} d(h,d) &\leq d(h',d') + (e',h')_h - (e',h)_{h'} \\ &\leq d(h',d') + 4\delta \\ &\leq R + 5\delta. \end{aligned}$$

But this implies that the geodesic from 1 to $h^{-1}e$ passes through C, and so $h^{-1}e \in E$ and $e \in hE$. Since e was arbitrary in $\partial G \setminus \partial P$, we have $PE = \partial G \setminus \partial P$, and so the action of P on $\partial G \setminus \partial P$ is cocompact. If $\bar{E}$ is the (compact) image of E in M, then $PE = M \setminus \{p\}$, and so p is a bounded parabolic point. $\square$

2.4. **Claim 4.**

Lemma 2.8. *For all $R > 0$ there is some D, depending only on R, G, $\mathcal{H}$, and S, so that for any g, $g' \in G$, and H, $H' \in \mathcal{H}$,*

$$\operatorname{diam}(N_R(gH) \cap N_R(g'H')) < D.$$

($N_R(Z)$ denotes the R-neighborhood of Z in the Cayley graph $\Gamma = \Gamma(G,S)$.)

Lemma 2.9. *There is some λ depending only on G, $\mathcal{H}$, and S, so that if x, $y \in gH \cup g\partial H$, then any geodesic from x to y lies in a λ-neighborhood of gH in Γ.*

Lemma 2.10. *Let $\gamma\colon \mathbb{R}_+ \to \Gamma$ be a (unit speed) geodesic ray, so that $x = \lim_{t\to\infty}\gamma(t)$ is not in the limit set of gH for any $g \in G$, $H \in \mathcal{H}$, and so that $\gamma(0) \in G$. Let $C > 0$. There is a sequence of integers $\{n_i\}$ tending to infinity, and a constant χ, so that the following holds, for all $i \in \mathbb{N}$: If $x_i = \gamma(n_i) \in N_C(gH)$ for $g \in G$ and $H \in \mathcal{H}$, then*

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([n_i,\infty))\right) < \chi$$

Proof. Let λ be the quasi-convexity constant from Lemma 2.9. Let D be the constant obtained from Lemma 2.8, setting $R = C + \lambda + 2\delta$, and let $\chi = 2D$.

Let $i \in \mathbb{N}$. If $i = 1$, let $t = 0$; otherwise set $t = n_{i-1} + 1$. We will find $n_i \geq t$ satisfying the condition in the statement.

If we can't use $n_i = t_0$, then there must be some gH with $g \in G$ and $H \in \mathcal{H}$ satisfying $\gamma(t_0) \in N_C(gH)$ and

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([t_0,\infty))\right) \geq \chi.$$

Let $s = \sup\{t \mid \gamma(t) \in N_C(gH)\}$. We claim that we can choose

$$n_i = s - \frac{\chi}{2} = s - (D + 4\delta + 2\lambda + 2C).$$

Clearly we have

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([n_i,\infty))\right) < \chi.$$

Now suppose that some other $g'H'$ satisfies $x_i = \gamma(n_i) \in N_C(g'H')$ and

$$\operatorname{diam}\left(N_C(g'H') \cap \gamma([n_i,\infty))\right) \geq \chi.$$

It follows (once one draws the picture) that $\gamma(n_i)$ and $\gamma(s)$ lie both in the $C + \lambda + 2\delta$ neighborhood of gH and in the $C + \lambda + 2\delta$ neighborhood of $g'H'$. Since $d(\gamma(n_i), \gamma(s)) = s - n_i = D$, this contradicts Lemma 2.8. □

Proof of Claim 4. Let $x \in \partial G \setminus \cup A$. We must show that $\bar{x} \in M$ is a conical limit point for the action of G on M. Fix some $y \in M \setminus \{x\}$, and let γ be a geodesic from y to x in Γ. Let $C = \lambda + 6\delta$, where λ is the constant from Lemma 2.9. Using Lemma 2.10, we can choose a sequence of (inverses of) group elements $\{{x_i}^{-1}\}$ in the image of γ so that whenever $x_i \in N_C(gH)$ for some $g \in G$, $H \in \mathcal{H}$, and $i \in \mathbb{N}$, we have

$$\operatorname{diam}\left(N_C(gH) \cap \gamma([n_i,\infty))\right) < \chi, \tag{3}$$

for some constant χ independent of g, H, and i.

Now consider the geodesics $x_i\gamma$. They all pass through 1, so we may pick a subsequence $\{x_i'\}$ so that the geodesics $x_i'\gamma$ converge setwise to a geodesic σ running from b to a for some $b, a \in \partial G$. In fact this sequence $\{x_i'\}$ will satisfy $\lim_{i\to\infty} x_i'x = a$ and $\lim_{i\to\infty} x_i'y' = b$ for all $y' \in \partial G \setminus \{x\}$. We will be able to use this sequence to see that $\bar{x}$ is a conical limit point for the action of G on M, *unless* we have $\bar{a} = \bar{b}$ in M.

By way of contradiction, we therefore assume that a and b both lie in $g\partial H$ for some $g \in G$, and $H \in \mathcal{H}$. The geodesic σ lies in a λ-neighborhood of gH, by Lemma 2.9. Let $R > \chi$. The set $x_i\gamma \cap B_R(1)$ must eventually be constant, equal to $\sigma_R := \sigma \cap B_R(1)$. Now σ_R a geodesic segment of length $2R$ lying entirely inside $N_C(gH)$. It follows that, for sufficiently large i, ${x'_i}^{-1}\sigma_R \subseteq \gamma$ lies inside $N_C({x'_i}^{-1}gH)$. In particular, if ${x'_i}^{-1} = \gamma(t_i)$, then we have $\gamma([t_i, t_i + R)) \subseteq N_C({x'_i}^{-1}gH)$. $R > \chi$, this contradicts (3). $\square$

References

[BM91] Mladen Bestvina and Geoffrey Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, 4(3):469–481, 1991.

[Bow98] Brian H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.

[Bow99] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012. Based on the 1999 preprint.

[Dah03] François Dahmani. Combination of convergence groups. *Geom. Topol.*, 7:933–963 (electronic), 2003.

[Ger12] Victor Gerasimov. Floyd maps for relatively hyperbolic groups. *Geom. Funct. Anal.*, 22(5):1361–1399, 2012.

[GP09] Victor Gerasimov and Leonid Potyagailo. Quasi-isometric maps and floyd boundaries of relatively hyperbolic groups. *arXiv preprint arXiv:0908.0705*, 2009.

[HY88] John G. Hocking and Gail S. Young. *Topology*. Dover Publications Inc., New York, second edition, 1988.

[Kap09] Michael Kapovich. Homological dimension and critical exponent of Kleinian groups. *Geom. Funct. Anal.*, 18(6):2017–2054, 2009.

[MOY12] Yoshifumi Matsuda, Shin-ichi Oguni, and Saeko Yamagata. Blowing up and down compacta with geometrically finite convergence actions of a group. *arXiv preprint arXiv:1201.6104*, 2012.

[Osi06] Denis V. Osin. Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. *Mem. Amer. Math. Soc.*, 179(843):vi+100, 2006.

[Tra13] Hung Cong Tran. Relations between various boundaries of relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 23(7):1551–1572, 2013.

[Yam04] AslıYaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.